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The oscillation of an infinite plate in a strong rarefied gas under constant force

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Abstract. The oscillation of an infinite plate in strong rarefied gas under constant force is investigated in the framework of the kinetic theory of gases. A model kinetic equation is solved using the method of moments, this method with two sided distribution function is used to replace a model equation describing the flow of the gas with non-linear moments equations.

The moments method, Laplace's transform and the small parameter method are used to solve this problem.

1. Introduction

The investigation of the behaviour of a rarefied gas near moving bodies is an aerodynamic problem of great interest. It is important to determine the rule governing the motion due to collision of the molecules with solid surfaces and collisions between the molecules themselves. Because of rarefaction of the gas there must be discontinuities in macroscopic parameters at the surfaces. In the last few years many authors, such as Grad (1949) and Weitzer (1965), have solved the problem of a half-space of gas bounded by a wall oscillating with small amplitude at a fixed frequency. Our aim in this paper is mainly to determine the velocity and shear stress of the plate, it is clear from the results that the solution is periodic. The distribution function is assumed to satisfy the Boltzmann equation, also we assume that the molecules are reflected from the surface diffusely with complete energy accomodation. The collision term is simplified by using the model suggested by Bhatnagar *et al* (1954).

2. Basic equations

Consider that the upper half of the space, which is bounded by an infinite plate $y = 0$, is filled with a highly rarefied gas. The plate is considered fixed and suddenly begins to oscillate itself in its own plane with velocity equal to $U \sin wt$ (U, w are constants). The plate is assumed impermeable and uncharged. The particles are moving under a constant field of forces $(0, ezE_0, 0)$ where E_0 is an external electric field and ez is the charge of each particle.

Assuming the gas to be highly rarefied, the induced electric field may be ignored. The distribution function $F(t, y, \bar{c})$ of the particles may be obtained from the kinetic equation:

$$\frac{\partial F}{\partial t}+c_y\frac{\partial F}{\partial y}+\frac{ezE_0}{m}\frac{\partial F}{\partial c_y}=0. \tag{2.1}$$

To solve (2.1), we shall use the solution in the form:

$$\mathrm{F}=\begin{cases} F_1=n_1(2\pi RT_1)^{-3/2}\exp[-(c_x-v_{x_1})^2+(c_1-v_{y_1})^2+c_z^2]/2RT_1 & c_y<0 \\ F_2=n_2(2\pi RT_2)^{-3/2}\exp[-(c_x-v_{x_2})^2+(c_y-v_{y_2})^2+c_z^2]/2RT_2 & c_y>0 \end{cases} \tag{2.2}$$

where n_1, n_2, T_1, T_2, v_{x_1}, v_{x_2}, v_{y_1}, and v_{y_2} are unknown functions of two variables y and t. Multiplying equation (2.1) by $\phi_i(c)$ and integrating over all values of $\bar{c}$ we get:

$$\frac{\partial}{\partial t}\overline{(\phi_i)}+\frac{\partial}{\partial y}\overline{(c_y\phi_i)}-\frac{ezE_0}{m}\overline{\left(\frac{\partial\phi_i}{\partial c_y}\right)}=0. \tag{2.3}$$

One can evaluate all mean quantities $\bar{g}$ by averaging over the velocity space:

$$\bar{g}=\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty}gF_1\,\mathrm{d}\bar{c}+\int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty}gF_2\,\mathrm{d}\bar{c}. \tag{2.4}$$

We introduce dimensionless variables by substitutions:

$$n_i=\bar{n}_i n_\infty, \qquad T_i=\bar{T}_i T_\infty, \qquad v_{x_i}=\bar{v}_{x_i}U,$$

$$v_{y_i}=\bar{v}_{y_i}U,\ w(i=1,2) \qquad y=\bar{y}U/w, \qquad t=\bar{t}/w.$$

If we take $M^2 \ll 1$ we can assume that the density and temperature variations at each point of the flow at any time are negligible, i.e. $\bar{n}_i=1+\mathrm{O}(M^2)$, $\bar{T}_i=1+\mathrm{O}(M^2)$ and $\bar{v}_{y_i}=\mathrm{O}(M^2)$.

Using the dimensionless variables and dropping terms of order $\mathrm{O}(M^2)$, we obtain from (2.3) for $\phi_1=c_x$ and $\phi_2=c_xc_y$

$$\begin{aligned} &\frac{\partial}{\partial t}(v_{x_1}+v_{x_2})+a_1\frac{\partial}{\partial y}(v_{x_2}-v_{x_1})=0 \\ &\frac{\partial}{\partial t}(v_{x_2}-v_{x_1})+a_2\frac{\partial}{\partial y}(v_{x_1}+v_{x_2})-\gamma_1(v_{x_1}+v_{x_2})=0. \end{aligned} \tag{2.5}$$

For simplicity of notation, we have dropped the 'bars' over the dimensionless variables. Also

$$\begin{aligned} &a_1=\frac{1}{M}(2/\pi\kappa)^{1/2}, \qquad && a_2=\frac{1}{M}(\pi/2\kappa)^{1/2}, \\ &\gamma_1=\frac{ezE_0}{mw}(2RT/\pi)^{1/2}, \qquad && M=U/(\kappa RT_\infty)^{1/2} \end{aligned} \tag{2.6}$$

and $\kappa=c_p/c_v$, is the ratio of specific heats.

The initial and boundary conditions are:

$$v_{x_1}(y,0)=v_{x_2}(y,0)=0, \tag{2.7}$$

$$\begin{aligned} &v_{x_2}(0,t)=\sin t \\ &v_{x_1} \text{ and } v_{x_2} \text{ are bounded at } y=\infty. \end{aligned} \tag{2.8}$$

Let $X = v_{x_1} + v_{x_2}$, $Y = v_{x_2} - v_{x_1}$ then equation (2.5) takes the form:

$$\begin{aligned} &\frac{\partial X}{\partial t} + a_1 \frac{\partial Y}{\partial y} = 0 \\ &\frac{\partial Y}{\partial y} + a_2 \frac{\partial X}{\partial y} - \gamma_1 X = 0. \end{aligned} \tag{2.10}$$

The initial and boundary conditions (2.7) and (2.8) become:

$$X(y, 0) = Y(y, 0) = 0 \tag{2.11}$$

$$\begin{aligned} &\tfrac{1}{2}[X(0, t) + Y(0, t)] = \sin t \\ &X,\ Y \text{ are bounded at } y = \infty. \end{aligned} \tag{2.12}$$

3. Solution of the governing equations

3.1. In the absence of an electric field

In this case equations (2.10) take the form

$$\begin{aligned} &\frac{\partial X}{\partial t} + a_1 \frac{\partial Y}{\partial y} = 0 \\ &\frac{\partial Y}{\partial t} + a_2 \frac{\partial X}{\partial y} = 0. \end{aligned} \tag{3.1}$$

This is a linear system of homogeneous partial differential equations with constant coefficients. By using Laplace transforms, and making the necessary algebraic manipulations, the solution of (3.1) may be written in the form El Safty (1977):

$$X^{(0)} = \begin{cases} 2 \sin(t - y/\sqrt{a_1 a_2})/(1 + \sqrt{a_1/a_2}) & t > y/\sqrt{a_1 a_2} \\ 0 & t \leq y/\sqrt{a_1 a_2} \end{cases} \tag{3.2}$$

$$Y^{(0)} = \begin{cases} 2\sqrt{a_2/a_1} \sin(t - y\sqrt{a_1 a_2})/(1 + \sqrt{a_2/a_1}) & t > y/\sqrt{a_1 a_2} \\ 0 & t \leq y/\sqrt{a_1 a_2}. \end{cases} \tag{3.3}$$

3.2. Using the small parameter method

Using the small parameter method and considering the electric field a small parameter, the solution can be put as:

$$\begin{aligned} X &= X^{(0)} + \gamma_1 X^{(1)} + \gamma_1^2 X^{(2)} \\ Y &= Y^{(0)} + \gamma_1 Y^{(1)} + \gamma_1^2 Y^{(2)}. \end{aligned} \tag{3.4}$$

By neglecting all terms of order $O(\gamma_1^3)$ we get $X^{(1)}$, $Y^{(1)}$, $X^{(2)}$, and $Y^{(2)}$ in the form El Safty (1977):

$$X^{(1)} = \begin{cases} \dfrac{y \sin(t - y/\sqrt{a_1 a_2})}{a_2(1 + \sqrt{a_2/a_1})} + \dfrac{\cos(t - y/\sqrt{a_1 a_2})}{(1 + \sqrt{a_2/a_1})^2} & t > \dfrac{y}{\sqrt{a_1 a_2}} \\ 0 & t \leq \dfrac{y}{\sqrt{a_1 a_2}} \end{cases} \tag{3.5}$$

$$Y^{(1)}=\begin{cases}\dfrac{y\sin(t-y/\sqrt{a_1a_2})}{\sqrt{a_1a_2}\,(1+\sqrt{a_2/a_1})}-\dfrac{\cos(t-y/\sqrt{a_1a_2})}{(1+\sqrt{a_2/a_1})^2} & t>\dfrac{y}{\sqrt{a_1a_2}}\\ 0 & t\leqslant\dfrac{y}{\sqrt{a_1a_2}}.\end{cases} \tag{3.6}$$

3.3. The solution when a fraction of molecules is reflecting specularly

Consider the case when a fraction θ (called the reflection coefficient) of the molecules is reflecting diffusely and the remaining fraction $(1-\theta)$ is reflecting specularly.

In this case the first boundary condition (2.8) takes the form:

$$v_{x_2}(0,t)=(1-\theta)v_{x_1}(0,t)+\theta\cos t. \tag{3.7}$$

By the same technique used above, we obtain in the absence of a magnetic field (Lees 1965):

$$X^{(0)}=\begin{cases}\dfrac{2\theta}{(\theta+(2-\theta)\sqrt{a_2/a_1})}\sin(t-y/\sqrt{a_1a_2}) & t>y/\sqrt{a_1a_2}\\ \pm 0 & t\leqslant y/\sqrt{a_1a_2}\end{cases} \tag{3.8}$$

$$Y^{(0)}=\begin{cases}\dfrac{2\theta\sqrt{a_2/a_1}}{(\theta+(2-\theta)\sqrt{a_2/a_1})}\sin(t-y/\sqrt{a_1a_2}) & t>y/\sqrt{a_1a_2}\\ 0 & t\leqslant y/\sqrt{a_1a_2}\end{cases} \tag{3.9}$$

$$X^{(1)}=\begin{cases}\dfrac{\theta y}{a_2(\theta+(2-\theta)\sqrt{a_2/a_1})}\sin(t-y/\sqrt{a_1a_2}) & \\ +\dfrac{\theta(2-\theta)}{(\theta+(2-\theta)\sqrt{a_2/a_1})^2}\cos(t-y/\sqrt{a_1a_2}) & \dfrac{t>y}{\sqrt{a_1a_2}}\\ 0 & \dfrac{t\leqslant y}{\sqrt{a_1a_2}}\end{cases} \tag{3.10}$$

$$Y^{(1)}=\begin{cases}\dfrac{\theta y}{\sqrt{a_1a_2}\,(\theta+(2-\theta)\sqrt{a_2/a_1})}\sin(t-y/\sqrt{a_1a_2}) & \\ -\dfrac{\theta^2}{(\theta+(2-\theta)\sqrt{a_2/a_1})}\cos(t-y/\sqrt{a_1a_2}) & \dfrac{t>y}{\sqrt{a_1a_2}}\\ 0 & \dfrac{t\leqslant y}{\sqrt{a_1a_2}}.\end{cases} \tag{3.11}$$

3.4. Solution of the problem for any γ_1

In this section we solve equation (2.10) with the initial and boundary conditions (2.11) and (2.12) for any γ_1.

By the same method used before (El Safty 1977) we can get the solution for X and Y in the form:

$$X=\begin{cases}A_1\sin(t-Ny/a_2+\epsilon_1) & t>Ny/a_2\\ 0 & t\leqslant Ny/a_2\end{cases} \tag{3.12}$$

$$Y = \begin{cases} B_1 \sin(t - Ny/a_2 - \epsilon_2) & t > Ny/a_2 \\ 0 & t \leqslant Ny/a_2 \end{cases} \tag{3.13}$$

where

$$A_1 = \frac{\exp(\gamma_1 y/2a_2)}{L^2 + \gamma_1^2/4}[\gamma_1^2(L-N)^2 + 4((\gamma_1^2/4) + LN)^2]^{1/2}$$

$$L = a_2/a_1 + N, \qquad N = [(a_2/a_1) - \gamma_1^2/4]^{1/2}$$

$$B_1 = \frac{\exp(\gamma_1 y/2a_2)(L-N)}{L^2 + \gamma_1^2/4}(\gamma_1^2 + 4L^2)^{1/2}, \qquad \epsilon_2 = \tan^{-1}\frac{\gamma_1}{2L}$$

$$\epsilon_1 = \tan^{-1}[\gamma_1(L-N)/2((\gamma_1^2/4) + LN)].$$

Also the solution when a fraction of molecules is reflecting specularly takes the form:

$$X = \begin{cases} A_2 \sin(t - Ny/a_2 + \Delta_1) & t > Ny/a_2 \\ 0 & t \leqslant Ny/a_2 \end{cases} \tag{3.14}$$

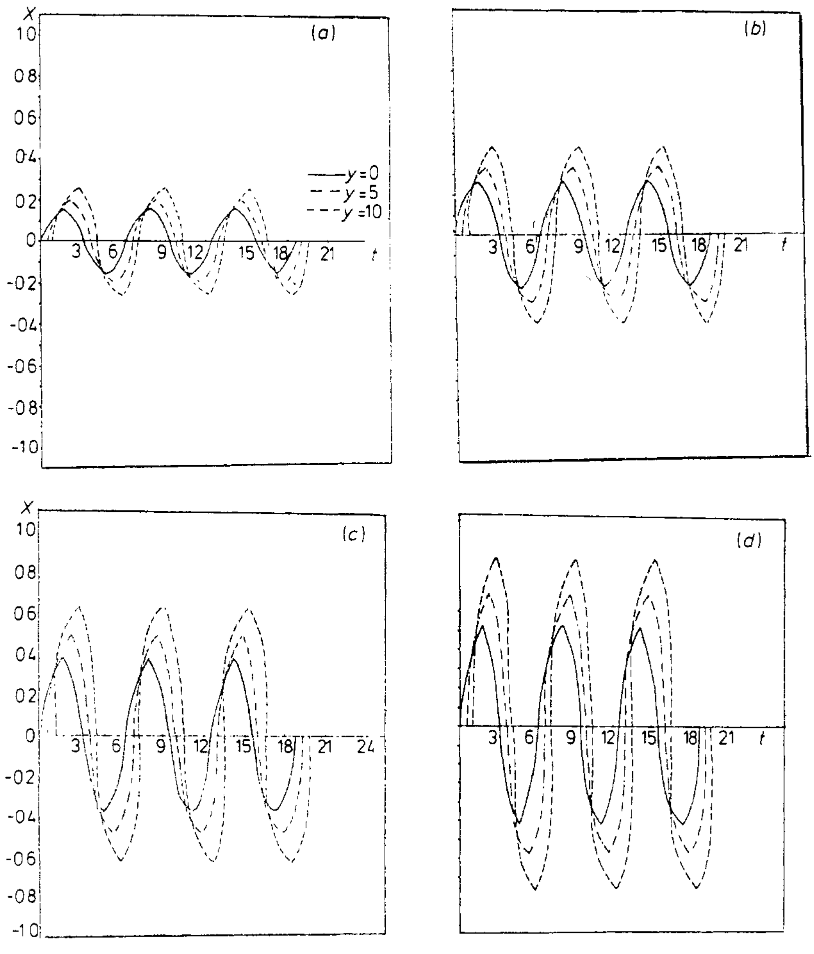

Figure 1. The relation between velocity (X) and time (t) for different distances (y) from the plate. $M = 0{\cdot}1$, $\gamma_1 = 1$. (a) Reflection coefficient $\theta = 0{\cdot}4$, (b) $\theta = 0{\cdot}6$, (c) $\theta = 0{\cdot}8$, (d) $\theta = 1$.

$$Y=\begin{cases} B_2\sin(t-Ny/a_2-\Delta_2) & t>Ny/a_2 \\ 0 & t\leqslant Ny/a_2 \end{cases} \tag{3.15}$$

where

$$A_2=\frac{\exp(\gamma_1 y/2a_2)}{L'^2+\gamma_1^2/4}[\gamma_1^2(L'-N)^2+4(\gamma_1^2/4+L'N)]^{1/2},$$

$$B_2=\frac{\theta(L'-N)\exp(\gamma_1 y/2a_2)}{(2-\theta)(L'^2+\gamma_1^2/4)}(\gamma_1^2+4L'^2)^{1/2}, \qquad \Delta_2=\tan^{-1}\gamma_1/2L',$$

$$\Delta_1=\frac{\tan^{-1}\gamma_1(L'-N)}{2(\gamma_1^2/4+L'N)}, \qquad L'=(2-\theta)a_2/a_1\theta+N$$

$$N=[(a_2/a_1)-\gamma_1^2/4]^{1/2}.$$

4. Discussion and conclusions

Numerical computations have been done to examine the behaviour of the velocity and shear stress.

The results are given in the following figures. (i) Figure 1 gives the relation between the velocity (X) when $\theta=0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, 1; $\gamma_1=1$; $M=0{\cdot}1$ and time (t) for different values of $y(y=0, 5, 10)$.

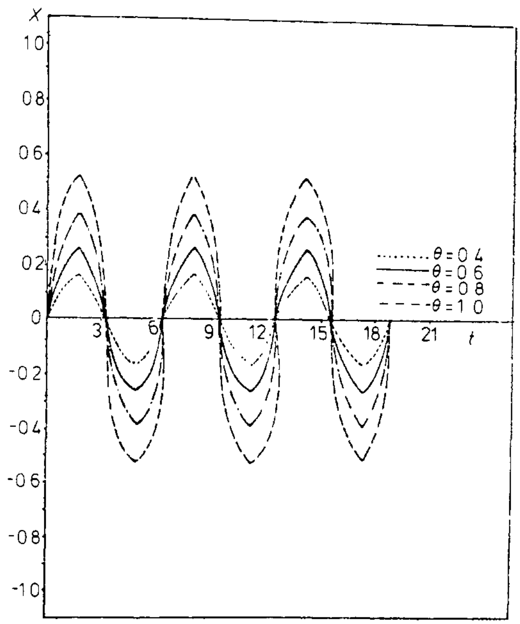

Figure 2. The relation between velocity (X) and time (t) for different reflection coefficients at the plate ($y=0$). $M=0{\cdot}1$, $\gamma_1=1$.

It is clear that: (*a*) for constant θ the velocity amplitude increases with y; (*b*) for constant y the velocity amplitude increases with θ; (*c*) there exists a phase angle velocity with the velocity of the plate which depends on y, θ, γ_1 and M. (ii) Figure 2 gives the velocity (X) at $y = 0$ and time (t) for different θ ($\theta = 0{\cdot}4$, $0{\cdot}6$, $0{\cdot}8$, 1). The slip velocity ($X = X(0) - \sin t$) is a periodic function whose amplitude is an increasing function of θ.

(iii) Figure 3 gives the relation between the shear stress (Y) for $\theta = 0{\cdot}4, 0{\cdot}6, 0{\cdot}8, 1$; $\gamma_1 = 1$; $M = 0{\cdot}1$ and time (t) for different values of $y (y = 0, 5, 10)$. These figures show that: (*a*) the amplitude of the shear stress is independent of time t, an increasing function of y and a decreasing function of θ; (*b*) the phase angle of the shear stress is also a function of y, θ, γ_1 and M.

(iv) In figure 4 the shear-stress jump at the surface of the plate is a periodic function in time and its amplitude decreases slowly with large variations of θ.

(v) The amplitudes of the velocity and shear stress increase with the increase of γ_1.

(vi) The periodic solution takes place only for values of $|\gamma_1| \leqq \sqrt{2\pi}$.

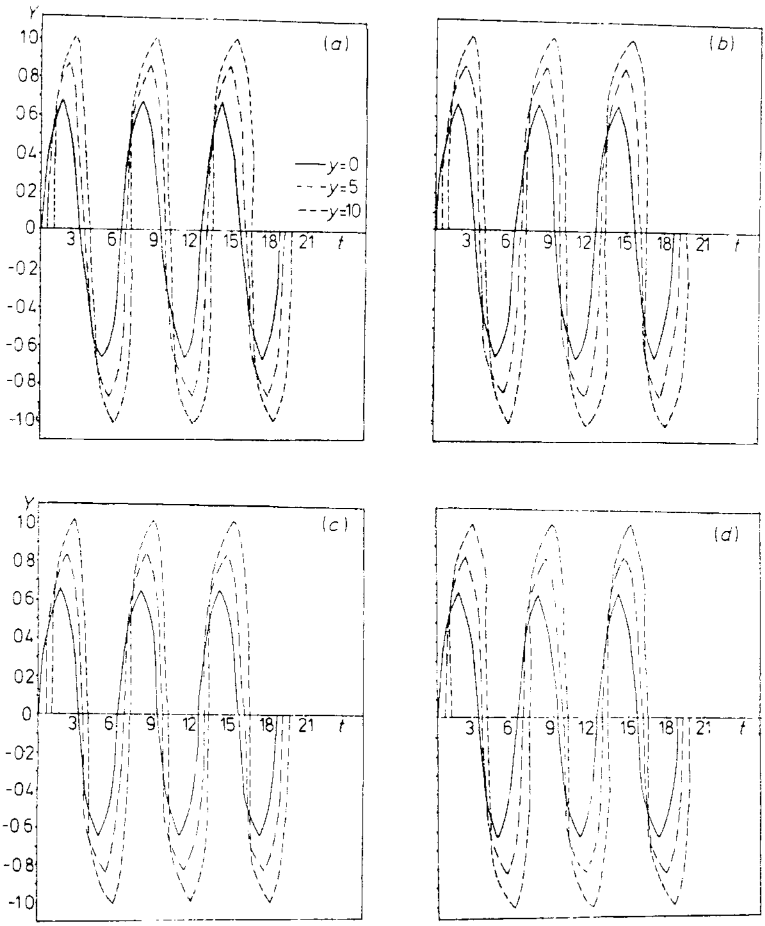

Figure 3. The relation between shear stress (Y) and time (t) for different distances from the plate for several reflection coefficients. $M = 0{\cdot}1$, $\gamma = 1$, (*a*) $\theta = 0{\cdot}4$, (*b*) $\theta = 0{\cdot}6$, (*c*) $\theta = 0{\cdot}8$, (*d*) $\theta = 1$.

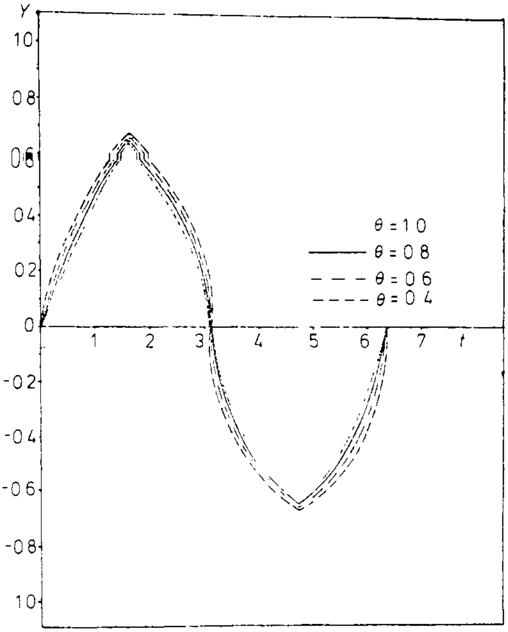

Figure 4. The relation between shear stress (Y) and time (t) for different reflection coefficients at the plate ($y = 0$). $M = 0{\cdot}1$, $\gamma_1 = 1$.

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